

EXERCICI RESOLT

Calcula les integrals següents:

a) $\int \frac{x^4 - 3x^2 + 2x - 2}{x - 1} dx$

b) $\int \frac{\sqrt[3]{x} + \sqrt{2x^2}}{\sqrt[4]{x}} dx$

c) $\int \frac{2\sin x + 3e^x}{5} dx$

RESOLUCIÓ

a) Fem la divisió i expressem el resultat de la forma $\frac{D}{d} = C + \frac{R}{d}$:

$$\int \frac{x^4 - 3x^2 + 2x - 2}{x - 1} dx = \int \left(x^3 + x^2 - 2x + \frac{-2}{x - 1} \right) dx = \frac{x^4}{4} + \frac{x^3}{3} - x^2 - 2\ln|x - 1| + k$$

$$\begin{aligned} \text{b) } \int \frac{\sqrt[3]{x} + \sqrt{2x^2}}{\sqrt[4]{x}} dx &= \int \left(\frac{\sqrt[3]{x}}{\sqrt[4]{x}} + \frac{\sqrt{2x^2}}{\sqrt[4]{x}} \right) dx = \int \left(x^{\frac{1}{3} - \frac{1}{4}} + \sqrt{2} x^{1 - \frac{1}{4}} \right) dx = \\ &= \int \left(x^{\frac{1}{12}} + \sqrt{2} x^{\frac{3}{4}} \right) dx = \frac{x^{13/12}}{13/12} + \sqrt{2} \frac{x^{7/4}}{7/4} + k = \frac{12}{13} \sqrt[12]{x^{13}} + \frac{4\sqrt{2}}{7} \sqrt[4]{x^7} + k \end{aligned}$$

$$\text{c) } \int \frac{2\sin x + 3e^x}{5} dx = \frac{-2\cos x + 3e^x}{5} + k$$

Calcula les integrals següents:

1 a) $\int (x^4 - 3x^2 + 2x - 1) dx$

b) $\int (x^3 - 2x) dx$

2 a) $\int \left(\frac{3}{4}x^5 - \frac{2}{3}x^2 + \frac{1}{7} \right) dx$

b) $\int x^3(x + 5) dx = \int (x^4 + 5x^3) dx$

3 a) $\int (x - 2)(x^2 + 4x) dx$

b) $\int (x^2 - 3)^2 dx = \int (x^4 - 6x^2 + 9) dx$

4 a) $\int (2x^2 - 3)^2 dx$

b) $\int \frac{\sqrt[5]{x} - \sqrt[3]{x^2}}{\sqrt{x}} dx$

5 a) $\int \frac{x^2 - \sqrt{x}}{2x} dx$

b) $\int \left(\frac{1}{2}e^x - \frac{3}{4}\cos x \right) dx$

6 a) $\int \frac{3}{\cos^2 x} dx$

b) $\int \frac{3\sin x + 2^x}{4} dx$

7 a) $\int \frac{3}{1 + x^2} dx$

b) $\int \frac{e^x + \operatorname{tg} x}{3} dx$

8 a) $\int \left(\frac{3}{x} + \frac{x^3}{3} + \frac{2}{x^4} \right) dx$

b) $\int \left(3^x + \frac{1}{x^3} - \frac{2}{x} \right) dx$

EXERCICI RESOLT

Calcula aquestes integrals:

a) $\int x\sqrt{1+x^2} dx$

b) $\int \frac{2 dx}{x \ln x}$

c) $\int \frac{5x+3}{x^2+1} dx$

RESOLUCIÓ

a) $\int x\sqrt{1+x^2} dx = \frac{1}{2} \int 2x(1+x^2)^{1/2} dx = \frac{1}{2} \frac{(1+x^2)^{3/2}}{3/2} + k = \frac{\sqrt{(1+x^2)^3}}{3} + k$

b) $\int \frac{2 dx}{x \ln x} = 2 \int \frac{1/x}{\ln x} dx = 2 \ln |\ln x| + k$

c) $\int \frac{5x+3}{x^2+1} dx = \int \left(\frac{5x}{x^2+1} + \frac{3}{x^2+1} \right) dx = \frac{5}{2} \int \frac{2x}{x^2+1} dx + 3 \int \frac{1}{1+x^2} dx =$
 $= \frac{5}{2} \ln |x^2+1| + 3 \arctan x + k$

Troba les integrals següents:

9 a) $\int 2x(x^2+1)^8 dx$

b) $\int x\sqrt{x^2-1} dx$

10 a) $\int x\sqrt{1-3x^2} dx$

b) $\int \cos x \cdot (\sin x)^5 dx$

11 a) $\int \sin x \cos^4 x dx$

b) $\int \frac{2x}{x^2+5} dx$

12 a) $\int \frac{x}{3x^2-2} dx$

b) $\int \frac{12x^2-4x}{4x^3-2x^2+1} dx$

13 a) $\int \frac{3x^2-x}{4x^3-2x^2+1} dx$

b) $\int \frac{\cos x}{\sin^6 x} dx$

14 a) $\int \frac{\cos x}{\sin x} dx$

b) $\int e^{6x+5} dx$

15 a) $\int 2x e^{x^2-3} dx$

b) $\int (x-1) e^{3x^2-6x} dx$

16 a) $\int \frac{e^{\lg x}}{\cos^2 x} dx$

b) $\int (6x^3-1) \cos(3x^4-2x) dx$

17 a) $\int (2^{5x} - \lg x) dx$

b) $\int 3 \cos x e^{\sin x} dx$

18 a) $\int \frac{e^{2x}}{e^{2x}+3} dx$

b) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$

Calcula les integrals següents:

19 a) $\int \frac{x + \sqrt{x}}{x} dx$

b) $\int x^4 (x + 1) dx$

20 a) $\int x^3 (2x^4 - 3)^5 dx$

b) $\int \frac{\sqrt[3]{3x} + \sqrt{2x}}{x} dx$

21 a) $\int \left(\frac{7x}{3} + \frac{3}{7x} \right) dx$

b) $\int \left(2x - \frac{1}{3} \right) (x^2 - 2) dx$

22 a) $\int \frac{3}{(x-2)^4} dx$

b) $\int (2x + 1)^3 dx$

23 a) $\int \sqrt[3]{(x+1)^2} dx$

b) $\int \cos x \sqrt{\sin x} dx$

24 a) $\int \frac{\cos x}{\sqrt[3]{\sin x}} dx$

b) $\int \frac{\cos x + \sin x}{\sin x - \cos x} dx$

25 a) $\int \frac{1 + \operatorname{tg}^2 x}{\operatorname{tg} x} dx$

b) $\int \frac{6x^3 - 1}{3x^4 - 2x + 1} dx$

26 a) $\int \frac{e^x + \cos x}{e^x + \sin x} dx$

b) $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$

27 a) $\int x \sin (x^2 - 2) dx$

b) $\int (3x^2 - 2) e^{x^3 - 2x} dx$

28 a) $\int \frac{e^{\ln x}}{x} dx$

b) $\int \frac{e^{\operatorname{arc} \operatorname{tg} x}}{1 + x^2} dx$

29 a) $\int x \cdot 3^{x^2 + 1} dx$

b) $\int e^x \sqrt{1 + e^x} dx$

30 a) $\int [\sin (3x + 1) + e^{3x}] dx$

b) $\int \frac{x}{(x^2 - 1)^3} dx$

31 a) $\int (2^{-x} + \ln x) dx$

b) $\int (\sqrt{x} + 2)(x + \sqrt{x}) dx$

32 a) $\int \frac{\operatorname{arc} \operatorname{tg} x}{1 + x^2} dx$

b) $\int \frac{x}{1 + x^4} dx = \int \frac{x}{1 + (x^2)^2} dx$

33 a) $\int \frac{\log_3 (x + 2)}{5} dx$

b) $\int \frac{\cos x}{1 + (\sin x)^2} dx$

34 a) $\int \frac{2}{\sqrt{1 - x^2}} dx$

b) $\int \frac{\pi}{1 + x^2} dx$

35 a) $\int \frac{x + 1}{x^2 + 1} dx = \int \left(\frac{x}{x^2 + 1} + \frac{1}{x^2 + 1} \right) dx$

b) $\int \frac{-3x}{\sqrt{1 - (x^2)^2}} dx$

2.2 MÈTODE DE SUBSTITUCIÓ

EXERCICI RESOLT

Resol, utilitzant un canvi de variable, les integrals següents:

a) $\int x^2 e^{x^3} dx$

b) $\int x \sqrt{x-1} dx$

c) $\int \frac{dx}{\sqrt{e^x - 1}}$

RESOLUCIÓ

a) En aquest cas, el canvi de variable el podríem haver realitzat mentalment; en les pàgines anteriors ja hem resolt directament integrals d'aquest tipus. Vegem com calcular-la mitjançant un canvi de variable:

Troblem $t = x^3 \rightarrow dt = 3x^2 dx$, queda així:

$$\int x^2 e^{x^3} dx = \frac{1}{3} \int e^{x^3} \cdot \underbrace{3x^2 dx}_{dt} = \frac{1}{3} \int e^t dt = \frac{1}{3} e^t + k = \frac{1}{3} e^{x^3} + k$$

b) Canvi: $t = \sqrt{x-1} \rightarrow t^2 = x-1 \rightarrow x = t^2 + 1 \rightarrow dx = 2t dt$

$$\begin{aligned} \int x \sqrt{x-1} dx &= \int (t^2 + 1) \cdot t \cdot 2t dt = \int (2t^4 + 2t^2) dt = \frac{2t^5}{5} + \frac{2t^3}{3} + k = \\ &= \frac{2\sqrt{(x-1)^5}}{5} + \frac{2\sqrt{(x-1)^3}}{3} + k \end{aligned}$$

c) Canvi: $t = \sqrt{e^x - 1} \rightarrow t^2 = e^x - 1 \rightarrow e^x = t^2 + 1 \rightarrow x = \ln(t^2 + 1) \rightarrow dx = \frac{2t}{t^2 + 1} dt$

$$\int \frac{dx}{\sqrt{e^x - 1}} = \int \frac{2t}{(t^2 + 1)t} dt = \int \frac{2t}{t^2 + 1} dt = 2 \operatorname{arc} \operatorname{tg} t + k = 2 \operatorname{arc} \operatorname{tg} \sqrt{e^x - 1} + k$$

1 Resol per substitució les integrals següents:

a) $\int \frac{dx}{1 + \sqrt[3]{x}}$ (Canvi: $t = \sqrt[3]{x}$)

b) $\int \frac{1+x}{1+\sqrt{x}} dx$

c) $\int \frac{x dx}{\sqrt{x+2}}$ (Canvi: $t = \sqrt{x+2}$)

d) $\int \frac{dx}{x\sqrt{2x-1}}$

e) $\int \frac{e^{2x}}{\sqrt{e^x + 1}} dx$ (Canvi: $t = \sqrt{e^x + 1}$)

f) $\int \frac{e^{2x} dx}{e^x + 1}$

g) $\int \frac{dx}{x\sqrt{x^2 - 1}}$ (Canvi: $t = \frac{1}{x}$)

h) $\int \sqrt{1-x^2} dx$ (Canvi: $x = \sin t$)

Utilitza la fórmula: $\cos^2 t = \frac{1}{2} + \frac{\cos 2t}{2}$

2.3 INTEGRACIÓ "PER PARTS"

FÓRMULA D'INTEGRACIÓ PER PARTS

$$\int u \, dv = uv - \int v \, du$$

EXERCICI RESOLT

Calcula les integrals següents:

a) $\int (2x + 1) e^x dx$

b) $\int e^x \cos x \, dx$

RESOLUCIÓ

a) Anomenem: $\begin{cases} u = 2x + 1 \rightarrow du = 2 \, dx \\ dv = e^x dx \rightarrow v = e^x \end{cases}$; i apliquem la fórmula:

$$\int (2x + 1) e^x dx = (2x + 1) e^x - \int 2e^x dx = (2x + 1) e^x - 2e^x + k = (2x - 1) e^x + k$$

b) ① $\begin{cases} u_1 = e^x \rightarrow du_1 = e^x dx \\ dv_1 = \cos x dx \rightarrow v_1 = \sin x \end{cases}$ ② $\begin{cases} u_2 = e^x \rightarrow du_2 = e^x dx \\ dv_2 = \sin x dx \rightarrow v_2 = -\cos x \end{cases}$

$$\underbrace{\int e^x \cos x dx}_I \stackrel{\textcircled{1}}{=} e^x \sin x - \int e^x \sin x dx \stackrel{\textcircled{2}}{=} e^x \sin x + e^x \cos x - \underbrace{\int e^x \cos x dx}_I$$

$$\Rightarrow 2I = e^x \sin x + e^x \cos x \Rightarrow I = \frac{e^x \sin x + e^x \cos x}{2} + k$$

1 Aplica la integració per parts per resoldre aquestes integrals:

a) $\int 3x e^x dx$

b) $\int (x^2 + 1) e^x dx$ (Aplica dues vegades el mètode.)

c) $\int (x + 1) \cos x dx$

d) $\int (x^2 + 2) \sin x dx$

e) $\int (x + 1)^2 e^x dx$

f) $\int x^2 \ln x dx$ ($u = \ln x$, $dv = x^2 dx$)

g) $\int \arctan x dx$ ($u = \arctan x$, $dv = dx$)

h) $\int e^x \sin x dx$